A New Perspective on Constrained Motion

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A new perspective on constrained motion

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The explicit general equations of motion for constrained discrete dynamical systems are obtained. These new equations lead to a simple and new fundamental view of lagrangian mechanics.

The principles of analytical mechanics laid down by D'Alembert, Lagrange (1787) and Gauss (1829) are all-encompassing, and therefore it naturally follows that there cannot be a new fundamental principle for the theory of motion and equilibrium of discrete, dynamical systems. Despite this, additional perspectives may yet be useful in understanding Nature's laws from new points of view, in particular if they can help in solving problems of special importance, and in providing deeper insights into the way Nature works.

While the general problem of constrained motion was formulated at least as far back as Lagrange, the determination of the explicit equations of motion for constrained, discrete dynamical systems, even within the restricted perview of langrangian mechanics, has been a major hurdle. The Lagrange multiplier method relies on problem-specific approaches to the determination of the multipliers; it is often very difficult to obtain them and hence to obtain the explicit equations of motion (both analytically and computationally) for systems which have a large number of degrees of freedom and many non-integrable constraints. Formulations offered by Gibbs, Volterra, Appell, and Boltzmann require a felicitous choice of problem-specific quasi-coordinates and suffer from similar problems in dealing with systems with large numbers of degrees of freedom and many non-integrable constraints.

In this paper we present the explicit, general equations of motion for constrained, discrete dynamical systems in terms of the generalized coordinates that describe their configurations. With the help of these new equations we then formulate a new fundamental principle of lagrangian mechanics.

Consider first an unconstrained, discrete dynamical system whose configuration is described by the n generalized coordinates $q := [q_1, q_2, q_3, \dots, q_n]^{\mathrm{T}}$. Its equations of motion can be described, using newtonian or lagrangian mechanics, by the relations

$$M(q,t)\,\ddot{q} = Q(q,\dot{q},t), \tag{1}$$

where the $n \times n$ matrix M is symmetric and positive definite. The generalized accelerations of the unconstrained system, which we denote by a, are thus given by

$$\ddot{q} = M^{-1}Q = a(q,\dot{q},t). \tag{2}$$

We now assume that the system is subjected to m consistent constraints (which need not be linearly independent) of the form

$$A(q,\dot{q},t)\,\ddot{q} = b(q,\dot{q},t), \tag{3}$$

where A is a known $m \times n$ matrix and b is a known m-vector. Differentiation of the usual constraint equations utilized in lagrangian mechanics, which are often in pfaffian form, will yield equations of the form (3). These constraint equations therefore include, among others, the usual holonomic, non-holonomic, scleronomic, rheonomic, catastatic and acatastatic varieties of constraints; combinations of such constraints may also be permitted in the equation set (3). We shall refer to the matrix A occurring in equation (3) as the *constraint matrix.* Thus equations (2), which describe the unconstrained system, and equations (3), which describe the constraints placed on this system, encompass *all* of lagrangian mechanics, and then some; for, the constraints (3) are more general than those that lie within the usual framework of lagrangian mechanics (Pars 1965).

The presence of the constraints (3), imposes additional 'generalized forces of constraint' on the system so that the explicit equations of motion of the constrained system now take the form

$$M\ddot{q} = Q(q,\dot{q},t) + Q_c(q,\dot{q},t), \tag{4}$$

where the additional term, Q_c, on the right-hand side arises by virtue of the imposed constraints which are proscribed by equations (3).

We begin by stating our result for the constrained system described above. For convenience, we state it in three equivalent forms.

1. The explicit equations of motion which govern the evolution of the constrained system are:

$$M\ddot{q} = Q + K(b - AM^{-1}Q), \tag{5a}$$

or

$$M\ddot{q} = Q + K(b - Aa), \tag{5a'}$$

where the matrix $K(q,\dot{q},t) = M^{\frac{1}{2}}\,(AM^{-\frac{1}{2}})^{+}$, and the superscript '+' denotes the Moore–Penrose generalized inverse (Moore 1920; Penrose, 1955) of the matrix, $AM^{-\frac{1}{2}}$.

2. The additional term on the right-hand side of equation (4) which represents the generalized force of constraint is explicitly given by

$$Q_c(q,\dot{q},t) = K(b - AM^{-1}Q). \tag{5b}$$

3. Equation (5a) can be rewritten after premultiplying by M^{-1} as,

$$\ddot{q} - a = M^{-1}K(b - Aa), \tag{5c}$$

or

$$\Delta a = K_1\, e, \tag{5c'}$$

where, the vector, $\Delta a = \ddot{q} - a$, represents the deviation (at the instant of time, t) of the constrained generalized acceleration, $\ddot{q}$, from the corresponding unconstrained acceleration, a; the error vector, $e = b - Aa$, represents the extent to which the accelerations, at the instant of time t, corresponding to the unconstrained motion do not satisfy the constraint equations (3); and, the matrix $K_1 = M^{-1}K = M^{-\frac{1}{2}}\,(AM^{-\frac{1}{2}})^{+}$.

In what follows, we shall refer to the matrix K_1 as the weighted Moore–Penrose generalized inverse of the weighted constraint matrix A.

The last form of our results lead to the following new fundamental principle of lagrangian mechanics:

The motion of a discrete dynamical system subjected to constraints evolves, at each instant of time, in such a way that the deviations of its accelerations from those it would have at that instant if there were no constraints on it, is directly proportional to the extent to which the accelerations corresponding to its unconstrained motion, at that instant, do not satisfy the constraints; the matrix of proportionality is the weighted Moore–Penrose generalized inverse of the weighted constraint matrix A, and the measure of the dissatisfaction of the constraints is provided by the vector e.

In more mathematical terms, the principle states that at each instant of time, the motion of a constrained, discrete dynamical system evolves so that the deviation Δa at each instant is directly proportional to e at that instant, the matrix of proportionality being K_1.

The derivation of this result is as follows.

Let us assume that at a given time t, $q(t)$ and $\dot{q}(t)$ are given. Then, Gauss's principle (Gauss 1829; Kalaba & Udwadia 1993) informs us that the accelerations, $\ddot{q}(t)$, are such that the Gaussian function, $\mathscr{G}$, defined as,

$$\mathscr{G} = [\ddot{q} - a(q, \dot{q}, t)]^{\mathrm{T}} M [\ddot{q} - a(q, \dot{q}, t)] \tag{6}$$

is minimized over all $\ddot{q}$ which satisfy the constraint equations (3). The (unique) solution to this constrained minimization problem yields the equations of motion.

Noting equations (3), and using the substitution $\ddot{r} = M^{\frac{1}{2}}\ddot{q}$, we have, according to the theory of generalized inverses,

$$\ddot{r} = (AM^{-\frac{1}{2}})^{+} b + Ry, \tag{7}$$

where y is an arbitrary vector, and R denotes the matrix $\{I - (AM^{-\frac{1}{2}})^{+}\,(AM^{-\frac{1}{2}})\}$. We determine the vector y by using Gauss's principle, which requires that

$$\mathscr{G} = \|\ddot{r} - M^{\frac{1}{2}}a\|_2^2 = \|Ry - \{M^{\frac{1}{2}}a - (AM^{-\frac{1}{2}})^{+} b\}\|_2^2 \tag{8}$$

be a minimum.

Since $R^{+} = R$, the vector y that minimizes $\mathscr{G}$ is given by

$$y = R\{M^{\frac{1}{2}}a - (AM^{-\frac{1}{2}})^{+} b\} + (AM^{-\frac{1}{2}})^{+}\,(AM^{-\frac{1}{2}})\,z, \tag{9}$$

where z is again an arbitrary vector. Using equation (9) in equation (7), result ($5a$) follows because $R \cdot R = R$ and $R(AM^{-\frac{1}{2}})^{+}\,(AM^{-\frac{1}{2}}) = 0$. □

We are led to marvel at the way Nature works; when the unconstrained motion of a system does not satisfy the constraints, Nature modifies the accelerations in a manner directly proportional to the extent to which these constraints are not satisfied, much like the calculating control theorist. The matrix of proportionality is, K_1, the weighted Moore–Penrose generalized inverse of the weighted constraint matrix. Little did Moore and Penrose realize at the time, that their invention of generalized inverses would play such a fundamental role in Nature's design; for, it is these seemingly abstract generalized inverses, that provide the key to understanding the complex interactions between impressed forces and the constraints.

The equations of motion obtained in this paper appear to be the simplest and most comprehensive so far discovered.

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